Tracing the Origin of the Mistaken Use of Versed Sine by Āryabhaṭa

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Abstract

Āryabhaṭa's use of versed sine functions in verses 35, 36 and 45 of Goḷapāda of Āryabhaṭīya had been a matter of conflict and discussions since the days of Brahmagupta down to the times of Bhāskara-II. Scholiasts of Āryabhaṭa school like Bhāskara-I and Lalla advocated the precepts of Āryabhaṭa while Brahmagupta and Bhāskara-II chose to correct them as they deemed fit according to the general astronomical canons. It is shown here that the Āryabhaṭa precepts on the two components of Valana, viz., Akṣa and Ayana which make use of the versed sine had their origin in the observations of the total lunar eclipse of 23 March 517 CE at Camravattam, 10N51, 75E45 – a village situated near the confluence of Bharatappuzha in Arabian Ocean. This lunar eclipse had presented Āryabhaṭa with the meridian transit of eclipsed Moon close to the equator which led him to the inference that formed the basis of the proportion made use of in the Akṣavalana rule – i.e. Versed sine of the hour angle and sine of Akṣavalana becoming zero simultaneously and akṣavalana attaining maximum on the horizon. Simultaneously the eclipse also presented Versed siine (Moon+90^0) equaling 1 and Ayanavalana attaining the maximum value of the obliquity of the earth's axis and thus leading to the proportion of verse 36 of Goḷapāda. Major reason that may have contributed to Āryabhaṭa's failure to have the precise formulation is the observation at the lower latitude of 10N51 and lack of additional observations to correct the deductions in verse 35, 36 and 45. Mistaken use of versed sine and the circumstances of the lunar eclipse of 23 March 517 CE when contrasted lead us to irrefutable evidence for the fact that Āryabhaṭa did witness the said eclipse before the formulation of Āryabhaṭīyam with epoch has Kali 3623(elapsed) or 522 CE, as accepted by the well known Haridatta tradition of Kerala.

Certain mistakes alleged to Āryabhaṭa by later astronomers like Brahmagupta could be explained as due to specific observations and inferences drawn by Āryabhaṭa in recent studies. In the same class of the omissions by Āryabhaṭa, we meet with the verses 35, 36 and 45 of Goḷapāda of Āryabhaṭīya where in Akṣa-valana (denoted by **A**) and Ayana-valana ($\mathbf{A_y}$) are discussed. Direction of the ecliptic in which the eclipsed moon and sun moves at a given instant is defined in terms of –

1. The angle between the east point on the equator and that on the prime vertical and
2. The angle between the east point on the ecliptic and on the equator.

Angle between the east point on the equator and the same on the prime vertical shall be decided by the latitude of observation and hence it received the name Akṣa-valana while the angle between east points on the ecliptic and equator is decided by the north-south declination or ayana of sun and therefore received the name Ayana-valana.

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Intricate geometry and trigonometry involved in computing the above angles during an eclipse finds detailed discussion with Burgess[1] and Shukla[2].

Rule of Aryabhata for the Latitudinal Correction (Akṣa Dṛk-karma)

(a) In Goḷapāda verse 35, Āryabhaṭa gives the rule as follows:

विक्षेपगुणाक्षज्या लम्बकभक्त्वा भवेत् ऋणमुदकस्थे।
उदये धनमस्तमये दक्षिणगे धनमृणं चन्द्रे ।। (35)

"Akṣajyā or Rsine of the local latitude ϕ multiplied by the latitude of moon and divided by Lambakam or Rcosine ϕ yields the value of akṣa-dṛk-karma i.e correction for the latitude effect. When moon is to the north of the ecliptic the correction is negative and for rising moon and positive for setting moon. When the moon is to the south of the ecliptic, it is positive for rising moon and negative for the setting moon"

In verse 45 of Goḷapāda, *Āryabhaṭa further stipulates in brief that the Rsin of the hour angle of the eclipsed body has to be multiplied by the sine of the latitude and divided by the radius.*

मध्याह्नोत्क्रमगुणितोfक्षो दक्षिणतोfर्धविस्तरहृतो दिक्।
स्थित्यर्धाच्चर्केन्दोस्त्रिराशिसहितायनात् स्पर्शे ।। (45)

"Akṣavalana is obtained by multiplying the Utkramjyā (Rversed sine) of hour angle with Rsine of the latitude of the place and dividing by the radius…"

Bhāskara-I has discussed the precept in detail in his work *Āryabhaṭa Karmanibandha* which came to be in known in later times as Mahābhāskarīyam.

Āryabhaṭa Tradition as Recorded by Bhāskara-I

Mahābhāskarīyam V.42-44:

"Multiply the Rversed-sine of the asus intervening between midday and the tithi (i.e. the time of the first contact, the middle of the eclipse, or of the last contact by the Rsine of the local latitude and divide that by the radius. Reduce the resulting Rsine to the corresponding arc called akṣavalana…"

i.e. Rsine A = (Rversin H*Rsin ϕ)/R, where H denotes the hour angle and ϕ the local latitude.

Brahmagupta had criticized the use of the Rversine and had instead used Rsin H in his precept on akṣavalana. Shukla has discussed the topic in the context of the verses 35 and 45 of Goḷapāda and has stated that the Sūryasiddhānta of Pañcasiddhāntikā (505-550CE) has the same rule as we see with Brahmagupta (628CE). Sūryasiddhānta extant now too gives the same formula as of Brahmagupta and thus it becomes apparent that the rule Rsine H existed before Brahamgupta and Āryabhaṭa as well.

As an illustration of the computation, values of the latitude of moon and latitude β are contrasted (Table-1) for two place with latitudes $\phi = 23.85$ and $\phi = 10.85$ for the date of Lunar eclipse, 23 March 517 CE. High values that arise when β is high shall be of no use as the technique had application only in graphical representation of the eclipses when the luminaries moved over the ecliptic and moon had very small latitude.

Validity of the Āryabhaṭa approximation

Rule of Āryabhaṭa according to the verse 35 of Goḷapāda yields the algorithm $A = R*\text{Sin}\,\phi*\beta/(R*\text{Cos}\phi) = \beta*\text{Tan}\,\phi$ for Akṣa-valana. According to Brahmagupta (628CE), Akṣavalana is given by[3] the formula:

$$\text{Rsin A} = \text{Rsin } Z_m * \text{Rsin } \phi/\text{Radius of Day circle } (\approx R)$$

- where Z_m is the zenith distance of moon on the prime vertical and ϕ is the latitude of the place.

And Bhāskara-II gave the expression as –

Rsin A = Rsin {(H in ghaṭis*90)/Semiduration of night in ghaṭis} x Rsinϕ/Rcosδ.

In modern terminology, we can write: $\text{Sin A} = \text{Sin } Z_m * \text{Sin } \phi/\text{Cos } \delta$ where Cos δ is the radius of the day circle and δ the declination of sun.

The zenith distance of the moon on the prime vertical can be approximated as the Natakāla, the hour angle and the values of akṣavalana obtained for the lunar eclipse[4] of 23 March 517 CE is shown in column 4 of Table-1.

Table-1: Illustration of the Āryabhaṭa Rule A = β* Tan ϕ

Col. 1	Col.2	Col.3		Col.4	
Time 23 March 517 CE	Latitude β′	Akṣavalana (Āryabhaṭa)		Akṣavalana (Modern)	
		ϕ $23^0.85$	ϕ $10^0.85$	ϕ $23^0.85$	ϕ $10^0.85$
17:00	17′	7.7′	3.3′	23.2^0	10.6^0
18:00	14′	6.2′	2.7′	23.8^0	10.8^0
19:00	11′	4.7′	2.0′	22.9^0	10.4^0
20:00	7′	3.2′	1.4′	20.4^0	9.3^0
21:00	4′	1.7′	0.7′	16.6^0	7.7^0
22:00	0	0.2′	0.1′	11.9^0	5.5^0
23:00	-3′	-1.3′	-0.6′	6.4^0	3.0^0
24:00	-6′	-2.8′	-1.2′	0.6^0	0.3^0

Above discussed algorithm by Āryabhaṭa i.e. $A = R*\text{Sin}\,\phi*\beta/(R*\text{Cos}\phi) = \beta*\text{Tan}\,\phi$ given in verse 35 can be correct only for low latitudes of observation and during an eclipse when both ϕ and β shall be very low values at which some comparable results to that of the correct formulae may be obtained. Contrast between the thumb rule of Āryabhaṭa and the correct values as per modern computations brings out the fact that the correction is higher depending on the higher latitude of observation and thus astronomers observing the skies at higher latitudes were better placed to make precise formulation of the same.

Ridicule of the rule of R-versed sine H

Rule which make use of **R-versed sine H** as we see in the Āryabhaṭa tradition had been under ridicule since the time of Brahmagupta. Sāstry, TSK in his discussion ascribes the mistaken use of versine to some ancient astronomers in the following words:[5]

"In order to mark the points of first and last contacts at the exact positions as seen by the observer, the lay of the segment of the ecliptic where the moon is situated has to be fixed with reference to the east-west of the observer. Two corrections, one due to the latitude of the place called akṣavalana and the other due to moon's ayana called ayanavalana have to be applied to the east-west points…But the versine of the hour angle is used instead of its sine , a mistake of some ancient astronomers"

Bhāskara-II criticized Lalla for the same in the following words: [6]

"When the sun is in the zenith and the eclipse looks like a vertical circle, then the valana obviously looks on the horizon like the agrā corresponding to the sun's longitude increased by three signs. If you, O' friend, proficient in spherics, can find out the same from the Rversed-sine (of the sun's longitude increased by the three signs) then indeed I must admit the flawlessness of the formula for the valana as stated in the Śiṣyadhīvṛdhida…"

How could a proficient astronomer like Āryabhaṭa err so much as to define the akṣvalana in terms of Rversine H?

It is known from literature that the Sūryasiddhānta rule of Rsine H may have existed even before Āryabhaṭa and thus it becomes a matter of curiosity as to what kind of observation may have misled Āryabhaṭa in framing the rule incorporating Rversine H?

Modification of Rsine H to Rversed sine H by Eclipse Observation

We noted above that the akṣa valana is of very small magnitude in the case of places with low latitudes and during eclipses when the latitude of the moon is small. Therefore accurate observation of the correction from the smaller latitudes of Kerala had been difficult for Āryabhaṭa and so it is likely that the original rule of Sūryasiddhānta applicable to north Indian latitudes got modified by replacing Rsine H by Rversed sine H.

The Rversed sine H rule instead of the Rsine H rule of Sūryasiddhānta by Āryabhaṭa can only be the result of inference drawn out of some critical observation[7]. Shukla has discussed a hypothetical situation in the discussion of the verses 42-44 of Mahābhāskarīya quoted earlier and in fact the hypothetical situation had been a real observation for Āryabhaṭa during the lunar eclipse of 23 March 517 CE as illustrated below.

Against the above background when we look at the lunar eclipse of 23 March 517CE, the following data (Table-2) strikes our attention.

Table-2: Circumstances of the Lunar Eclipse of 23 March 517 CE

Time	Sun λ^0	Moon λ^0	Versine	Mid-heaven	H hours	Akṣa-Valana^0		
						Āryabhaṭa	B.Gupta	Modern
17:00	4.45^0	180.57^0	1.227	78.56^0	6.87	13.35^0	-16.72^0	10.53^0
18:00	4.49	181.18	0.976	92.33	5.91	10.58	-16.60	10.85
19:00	4.53	181.79	0.726	106.16	4.94	7.86	-15.39	10.48
20:00	4.57	182.40	0.494	120.33	3.97	5.34	-13.28	9.44
21:00	4.61	183.01	0.294	135.09	3.01	3.17	-10.47	7.79
22:00	4.66	183.62	0.139	150.55	2.04	1.50	-7.22	5.64
23:00	4.70	184.23	0.039	166.64	1.07	0.42	-3.77	3.14
0:00	**4.74**	**184.84**	**0.000**	**183.07**	**0.11**	0.00	-0.36	**0.69**
1:00	4.78	185.46	0.025	199.41	-0.86	0.27	2.79	2.46
2:00	4.82	186.07	0.112	215.29	-1.83	1.21	5.51	5.03
3:00	4.86	186.68	0.256	230.49	-2.79	2.76	7.64	7.31
4:00	4.90	187.30	0.447	245.01	-3.76	4.82	9.08	9.11
5:00	4.94	187.91	0.673	259.02	-4.73	7.28	9.80	10.32
6:00	4.98	188.53	0.920	272.78	-5.69	9.97	9.79	10.85^0

It may be noted that –

1. The lunar eclipse presented an occasion when the eclipsed moon had a meridian transit when posited close to the equator.
2. At the intersection of the meridian and the equator, the Rversed sine of the hour angle H was zero and the Rsine of the akṣavalana was also zero.
3. Rversed sine of the hour angle and the akṣavalana increased thereafter.
4. When the eclipsed body was at the horizon at 18:00 hrs on 23 March or at 06:00 hrs on 24 March, the Rversed sine of H had its maximum value and the akṣavalana also had its maximum value equal to the latitude of the place.

In view of the above observation, Āryabhaṭa did draw the inference that the Rsine of the akṣavalana varied as the Rversed sine of the hour angle and therefore the Rsine of the akṣavalana for any time may be found by making use of the proportion.

It may be noted that the Brahmagupta rule also gave high values and when the eclipse body was at the horizon, the rule of Āryabhaṭa viz., the Rversed sine rule gave the latitude more precisely than the Rsine H rule.

Fig.1 presents the functions versine H and sine H as a function of time or Moon’s longitude during the night of eclipse.

Fig.1: Versine and Sine During the Night of Lunar Eclipse

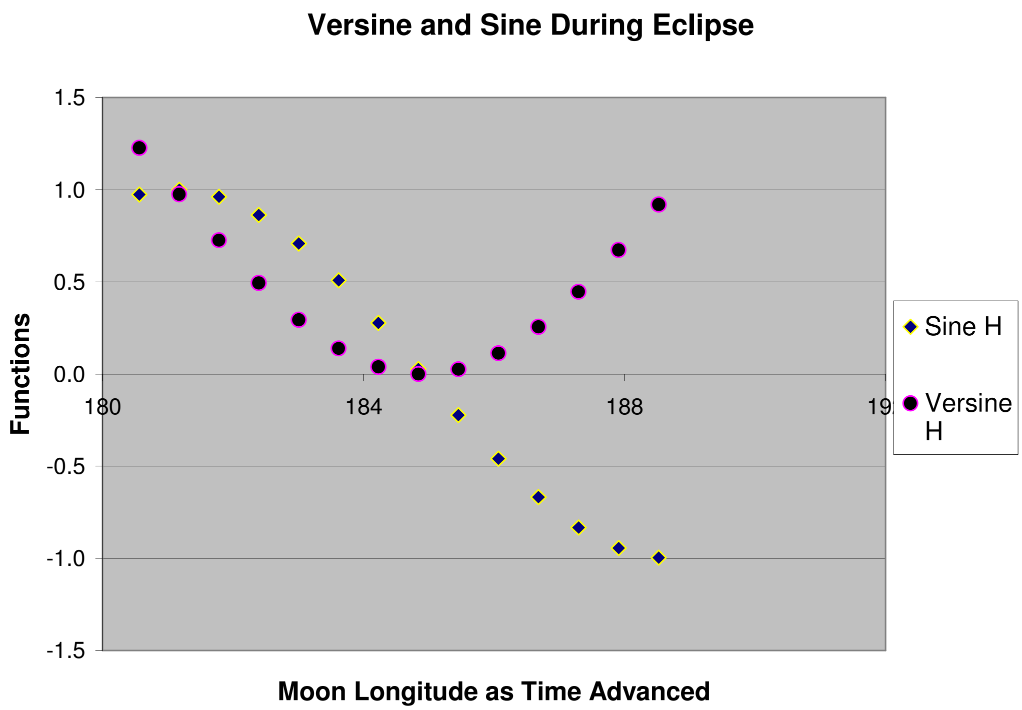


Fig.1 is illustrative of the fact that the function versine H was chosen by Āryabhaṭa to represent the eclipsed body at the horizon at 06:00 hrs on 24 March when the Rversed sine of H had its maximum value and the akṣavalana had to be equal to the latitude of the place. East to West horizon observations received the correct representation with versine H but the same failed for the intended purpose during the progress of the eclipse between first contact and last contact.

Use of Versine in Ayana-valana

Goḷapāda verse 36 defines Ayana-valana as –

विक्षेपापक्रमगुणमुत्क्रमणं विस्तरार्धकृतिभक्तम्।
उदगृणधनमुदगयने दक्षिणगे धनमृणं याम्ये।। (36)

“Rversed sine of the moon’s longitude (λ) increased by 90^0 multiplied by the Rsine of the obliquity (ω) and the latitude of the moon (β) gives the Ayana-valana (A_y)….”

i.e. $A_y = (\text{Rversin}\ (\lambda+90^0)*\text{Rsin}\ \omega*\beta)/R^2$

Brahmagupta on the other hand gave the rule as –

$$A_y = (R\sin(\lambda+90^0)*R\sin\omega*\beta)/R^2$$

A contrast of the functions Versine λ+90 and sine λ+90 is presented in Table-3 and the magnitudes of $\mathbf{A_y}$ for both are plotted in fig 2 for the night of the lunar eclipse of 23 March 517CE.

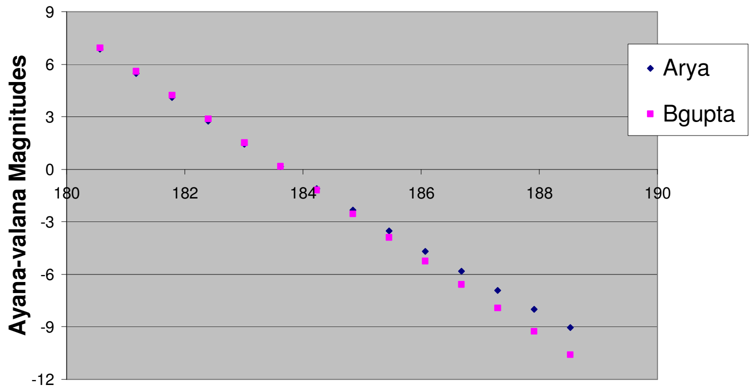
Table-3: A_y Results of Āryabhaṭa and Brahmagupta

Time	Moon λ^0	$\lambda+90^0$	Versine $\lambda+90^0$	Sine $\lambda+90^0$	Āryabhaṭa		Brahmagupta	
					A_y^0	$\beta*A_y^0$	A_y^0	$\beta*A_y^0$
17:00	180.57	270.57	0.99	-1.00	23.75	6.87	-24.00	-6.95
18:00	181.18	271.18	0.98	-1.00	23.48	5.47	-23.99	-5.59
19:00	181.79	271.79	0.97	-1.00	23.21	4.10	-23.99	-4.24
20:00	182.40	272.40	0.96	-1.00	22.94	2.76	-23.98	-2.88
21:00	183.01	273.01	0.95	-1.00	22.67	1.44	-23.96	-1.53
22:00	183.62	273.62	0.94	-1.00	22.40	0.16	-23.95	-0.17
23:00	184.23	274.23	0.93	-1.00	22.13	-1.10	-23.93	1.19
0:00	184.84	274.84	0.92	-1.00	21.86	-2.32	-23.91	2.54
1:00	185.46	275.46	0.90	-1.00	21.60	-3.52	-23.88	3.89
2:00	186.07	276.07	0.89	-0.99	21.33	-4.68	-23.86	5.24
3:00	186.68	276.68	0.88	-0.99	21.06	-5.82	-23.83	6.58
4:00	187.30	277.30	0.87	-0.99	20.80	-6.93	-23.79	7.92
5:00	187.91	277.91	0.86	-0.99	20.53	-8.00	-23.76	9.26
6:00	188.53	278.53	0.85	-0.99	20.27	-9.05	-23.72	10.59

It is evident from the above that in the case of Ayana-valana too Āryabhaṭa deduced the inference of Rversed sine inspired by the lunar eclipse observation of 23 March 517CE. Lunar eclipse of 23 March 517 CE presented him with the situation of Rversed sine λ+90 ≈ R so that the Ayana-valana is equal to the obliquity (ω) of the ecliptic and thus emerged the proportion that we find in his rule. Direction in the case of sine λ+90 differed and also magnitudinal changes. Comparison of the respective arcs obtained as $A_y*\beta$ in the case of Āryabhaṭa and Brahmagupta shows that with small values of β in an eclipse, the values of Ayana-valanas were very close to each other and thus Āryabhaṭa may have found the rule satisfactory for graphical representation of eclipses. Plot presents a comparison of the magnitudes for the lunar eclipse of the night of 23 March 517 CE.

It can easily be understood that between the first contact and the last contact, the Āryabhaṭa values of β*Ay were indistinguishably equal to that of Brahmagupta and thus it is clear that the precept of Āryabhaṭa which make use of the Rversed sine had their origin in astronomical -

Fig-2: Aryabhata and Brahmagupta:
Ayana Valana Magnitudes



- observations inspired by the utility in practical application. Lack of resources or time and convenience in ancient times prone to political instabilities may have prevented Āryabhaṭa and his direct disciples from carrying out additional observations and deduction of more general rules.

Conclusions

Discussion as above on the use of versed sine function by Āryabhaṭa in contrast to the circumstances of the total lunar eclipse of 23 March 517 CE renders us some new light on the nature of observations and the times that gave shape to the immortal work Āryabhaṭīyam. Present work is supportive of the earlier publications on the native place of Āryabhaṭa and the origin of Kerala school of Indian Astronomy and reinforces the Haridatta tradition that the epoch of Āryabhaṭīyam is Kali 3623 (elapsed) or 522CE.

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[5] Sāstry, TSK, *Historical Development of Hindu Astronomical Process*, Indian Journal of History of Science, May-November 1969, INSA, New Delhi-2, p.117

[6] Shukla, Kripa Shankar, Vaṭeśvarasiddhānta-II, Indian National Science Academy, New Delhi-2, (1981), p.455

[7] Shukla has discussed the hypothetical situation under which such a wrong inference could be drawn by an astronomer in his Mahābhāskarīyam translation (Ref.2), p.170.